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Sums of Perturbed Sequences of Integers

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An increasing sequence of nonnegative integers A is an asymptotic basis or a subbasis of order h if the sequence $A+A+\cdots+A$ (h summands) contains all sufficiently large integers or contains an infinite artihmetic progression, respectively. Such basis properties of a sequence A may be affected by small perturbations of the elements of A. The stability of these basis properties under small perturbations is examined in a general setting as well as for the specific sequence of the primes. The relation of the latter case to the Goldbach conjecture is noted. © 1988 Academic Press, Inc.

In many fields of mathematics, notably differential equations, stability of solutions is a major topic of interest. This is, of course, a problem not so much of "solutions" but a problem of examining the stability of properties. The concept of stability is usually associated with continuous mathematics and usually not considered in relation to "discrete" mathematics. It has occurred in the form of discrepancy in uniform distribution modulo 1 but this could also be argued to be analytic in nature.

An application of stability in discrete areas was due to Burr [1] where he examined the completeness properties of sequences of perturbed polynomial values. A sequence A is complete if all sufficiently large integers can be expressed as a finite sum of elements of A. Later, Burr and Erdös [2] extended this idea to more general sequences.

In the following article we intend to examine the stabiity of other properties of a sequence. In particular, we will be concerned with sequences that can represent all sufficiently large integers contained in a residue class modulo m. A sequence A is a (asymptotic) basis of order h if all

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(sufficiently large) positive integers can be expressed as the sum at most h (not necessarily distinct) elements of A. We will denote this sum set by hA. The major distinction between a basis and a complete sequence is the limit on the number of summands available.

We now make the term "perturb" a precise concept.

DEFINITION. Let A be an increasing sequence of nonnegative integers. Let P be any set of integers. A sequence B is a P-perturbation of A if $b_n - a_n \in P$ for all n. If the set $P = \{0, 1, ..., k\}$ we will refer to a k-perturbation of A.

It is easily observed that a basis of any order is not a very stable concept. Indeed, a 1-perturbation can destroy the property. All that is required is to shift all the terms to the 0 residue class modulo 2. To avoid this trivial situation, we will define a sequence A to be a subbasis of order h if hA contains an infinite arithmetic progression, i.e., all sufficiently large elements in a residue class modulo m.

We intend to develop some of the fundamental properties of perturbed sequences as they relate to the property of being a subbasis. In conclusion, we will establish a rather surprising result that sheds some light on the inherent difficulty in solving Goldbach's conjecture.

In the following, all sequences will be increasing sequences of nonnegative integers. If a perturbing set P has negative terms we will still assume that all the terms of the perturbed sequence B are nonnegative.

The purpose of Lemma 1 is to allow us to assume that a k-perturbation of an increasing sequence can, without loss of generality, be assumed to be increasing also.

LEMMA 1. *Let A be an increasing sequence of nonnegative integers and let B be a k-perturbation of A. If a_i is perturbed to b_i and a_{i+1} is perturbed to b_{i+1} with $b_i \geqslant b_{i+1}$, then perturbing a_i to b_{i+1} and a_{i+1} to b_i is also a k-perturbation.*

Proof. We may assume $b_i \geqslant a_i$, $b_{i+1} \geqslant a_{i+1}$, and $a_{i+1} > a_i$. Thus $a_i < a_{i+1} \leqslant b_{i+1} \leqslant b_i$. Since B is a k-perturbation of A, $k \geqslant b_i - a_i \geqslant b_{i+1} - a_i \geqslant 0$. Thus we can perturb a_i to b_{i+1} with a k-perturbation. Similarly, $k \geqslant b_i - a_i \geqslant b_i - a_{i+1} \geqslant 0$ so that we can perturb a_{i+1} to b_i with a k-perturbation.

In the study of completeness properties by Erdös and Burr it was assumed that repetition of elements in a sequence would be allowed. In the case of a subbasis there is nothing to be gained by repetition of elements in a sequence or in its perturbation. Because of this we will view a perturbation of a sequence A into a sequence B as an onto mapping from A to B.

That is a P-perturbation is a function $f\colon A \to B$ satisfying $b_i - a_j \in P$ for all $a_j \in f^{-1}(b_i)$.

We begin with a result on the stability of $\mathbb{Z}_0 = \{0, 1, ...\}$ itself.

THEOREM 1. *Every* 1-*perturbation of* $\mathbb{Z}_0$ *is a subbasis of order* 2.

Proof. Let B be an arbitrary but fixed 1-perturbation of $\mathbb{Z}_0$. Consider an even integer $2n$, if $n \in B$ then $2n \in B + B$. If $n \notin B$ it follows that $n-1 \in B$ and $n+1 \in B$. Thus $2n \in B+B$. In any case, $\{2n\}_{n=1}^{\infty} \subset B+B$ so B is a subbasis of order 2.

The result of Theorem 1 is not particularly startling but it is the best possible result in the sense expressed by Theorem 2.

THEOREM 2. *Let A be any increasing sequence of nonnegative integers. There exists a* 2-*perturbation of A which is not a subbasis of order* 2.

Proof. We will show that there exists a 2-perturbation of $\mathbb{Z}_0$ which is not a subbasis of order 2.

Let $\{A_i\}_{i=0}^{\infty}$ be any enumeration of the countable set of all arithmetic progressions in $\mathbb{Z}_0$ where $1 \in A_0$. That is each A_i contains all the nonnegative terms of an arithmetic progression. We will show that there is a 2-perturbation B of $\mathbb{Z}_0$ such that $B+B$ does not represent arbitrarily large elements in each A_i.

Begin by listing the sets $\{A_i\}_{i=0}^{\infty}$ in the following order; A_0, A_1, A_0, A_1, A_2, A_0, A_1, A_2, A_3, A_0, ... and relabel them by A_0', A_1', A_2', ... It suffices to construct a sequence $\{n_i\}_{i=0}^{\infty}$ such that $n_0 = 1$, $n_i \in A_i'$ and $n_i \notin B+B$ for all i.

We begin by defining $b_0 = 0$, $b_1 = 2$ so that $n_0 = 1 \notin B+B$ and $n_0 \in A_0'$. We introduce some notation that will allow us to describe the perturbed sequence B via the gaps between the terms of the sequence. Define $g_j = b_j - b_{j-1}$, with $b_0 = 0$. Thus $b_1 = g_1 = 2$ and $b_k = \sum_{i=1}^{k} g_i = g(k)$ where $1 \leqslant g_i \leqslant 3$ for all i since B is a 2-perturbation of $\mathbb{Z}_0$. Denote partial sums of gaps by $g_j + g_{j+1} + \cdots + g_k = g(j, k)$. Thus $g(1, k) = g(k)$. It should be noted that $m \in B+B$ if and only if $m = b_j + b_k = 2g(j) + g(j+1, k)$ for some j, k, $j \leqslant k$.

Assume g_1 up to g_s have been chosen so that

(i) each $g_i = 2$ or 3,

(ii) $n_0, ..., n_r \notin B+B$ with $n_i \in A_i'$,

(iii) $g(s) > n_r$.

Now let $n_{r+1} \in A_{r+1}'$ be chosen such that $n_{r+1} > 2g(s)$. For convenience, let $n_{r+1} = n$. Choose $g_{s+1} = g_{s+2} = \cdots = g_t = 3$ where $2g(t) + i = n$ for some i, $1 \leqslant i \leqslant 6$.

Next we need to determine the gaps after $g(t)$. To determine g_{t+l}, let j satisfy $2g(j)+g(j+1, t+l-1)<n$ and $2g(j+1)+g(j+2, t+l-1)>n$.

Case 1. If $g_j=3$ and $g_{j+1}=2$, let $g_{t+l}=3$. Then $2g(j)+g(j+1, t+l)=2g(j)+g(j+1, t+l-1)+g_{t+l} = 2g(j)+g(j+1, t+l-1)+3>n$ while $2g(j-1)+g(j, t+l)=2g(j)+g(j+1, t+l-1)-g_j+g_{t+l}<n$. It is clear that we also have $2g(j-s)+g(j-s+1, t+l)<n$ for $s>1$.

Case 2. If $g_j=g_{j+1}=3$, let $g_{t+l}=3$. Then $2g(j)+g(j+1, t+l)=2g(j)+g(j+1, t+l-1)+g_{t+l}=2g(j)+g(j+1, t+l-1)+3>n$ while $2g(j-1)+g(j, t+l)=2g(j)+g(j, t+l-1)-g_j+g_{t+l}<n$. Again $2g(j-s)+g(j-s+1, t+l)<n$ for $s>1$.

Case 3. If $g_j=g_{j+1}=2$, let $g_{t+l}=2$. Then $2g(j)+g(j+1, t+l)=2g(j)+g(j+1, t+l-1)+g_{t+1}=2g(j)+g(j+1, t+l-1)+2>n$ while $2g(j-1)+g(j+1, t+l)=2g(j)+g(j+1, t+l-1)-g_j+g_{t+l}<n$. Again $2g(j-s)+g(j-s+1, t+l)<n$ for $s>1$.

Case 4. If $g_j=2$ and $g_{j+1}=3$. If $2g(j)+g(j+1, t+l-1)=n-1$, let $g_{t+l}=2$. Then $2g(j)+g(j+1, t+l)=2g(j)+g(j+1, t+l-1)+g_{t+l}=n+1$. If $2g(j)+g(j+1, t+l-1)=n-2$, let $g_{t+l}=3$. Then $2g(j)+g(j+1, t+l)=n+1$ and $2g(j-1)+g(j, t+l)=n-1$.

The result of Theorem 2 can be improved slightly if certain growth conditions are assumed about the sequence A. (The proof technique is of particular importance. The ideas will play a key role in Theorem 5.)

THEOREM 3. *Let $S=\{s_n\}_{n=0}^{\infty}$ be a sequence such that for any $M\in\mathbb{Z}_0$ there exists an $N=N(M)$ such that $j>N$ implies $s_{j+1}-s_j>M$. Then there exists a $\{-1, 0\}$-perturbation of S which is not a subbasis of order* 2.

Proof. We proceed as in the proof of Theorem 2. We will construct a sequence $\{n_i\}_{i=1}^{\infty}$ with $n_i\in A_i'$ and $n_i\notin S'+S'$ where S' is the perturbation of S. We begin by choosing $n_1\in A_1'$ such that n_1, $n_1-1\notin S$. This can be done because of the gap conditions on S. Let $n_1=s_{i_1}+s_{i_1}'=\cdots=s_{i_t}+s_{i_t}'$ be all the ways n_1 is representable as a sum of two elements of S with $s_{i_j}\geqslant s_{i_j}'$. If there is no representation, no perturbation is needed. If there are such representations of n_1, perturb each s_{i_j}' to $s_{i_j}'-1$ to form S_1'.

Clearly n_1 is not the sum of two unperturbed elements of S_1'. It is also easily seen that n_1 is not the sum of two perturbed terms since each $s_{i_j}'\leqslant n_1/2$ so $(s_{i_j}'-1)+(s_{i_k}'-1)<n_1/2+n_1/2=n_1$. If $n_1=s_{i_j}+s_{i_k}'-1$ then $s_{i_j}\in S$ and $s_{i_j}-1\in S$ since we would also have $n_1=s_{i_k}+s_{i_k}'$ so $s_{i_k}+1=s_{i_j}$. But s_{i_k} and s_{i_j} are at least as large as $n_1/2$ and this violates the increasing gap sizes in S if n_1 is chosen large enough.

Choose n_k such that $n_k\in A_k'$, $n_k-j\notin S$ for $j=0, 1, ..., k$. Again we are assured that this can be done by the gap condition on S. In order to be sure we will not perturb any of the previously perturbed terms in S'_{k-1}, choose n_k large enough so that for $s_i \geqslant n_i/2$, $s_{i+1} - s_i > n_{k-1}$.

Continuing in this fashion we will have a sequence S', a $\{-1, 0\}$-perturbation of S, such that S' is not a subbasis of order 2.

Although, as pointed out earlier, a basis is a very unstable concept under perturbation, it is reasonable to ask whether sequences which are "close" to being a basis can be easily perturbed into a basis. The next theorem shows that in general this cannot be done. The density referred to is the lower asymptotic density. If A is an increasing sequence of nonnegative integers let $A(n) = \sum 1$ where $a_j \in A$, $1 \leqslant a_j \leqslant n$. Then $\mathbf{d}(A) = \underline{\lim}_{n \to \infty} (A(n)/n)$.

THEOREM 4. *Let n be a positive integer. There exists a sequence A such that $\mathbf{d}(A + A) = 1$ but no n-perturbation of A is an asymptotic basis of order* 2.

Proof. Let n be given and let $k = 2(n+2)$. Define $A = \{1, 2, ..., k, 2k, ..., k^2, (k+3)k, ..., k^3, (k+3)k^2, ..., k^4, (k+3)k^3, ...\}$. Then $\underline{\lim}_{n \to \infty} ((k+3)k^n - 2nk)/(k+3)k^n = 1 \leqslant \mathbf{d}(A + A)$.

Let $B = \{k^l + k + 2n + 1\}_{l=1}^{\infty}$. Then $B \cap (A + A) = \varnothing$, and it is easily seen that for any n-perturbation A' of A, $B \cap (A' + A') = \varnothing$ so A' cannot be an asymptotic basis of order 2.

We next examine how the concept of perturbation can relate to problems such as the Goldbach conjecture that every even integer greater than 4 is expressible as a sum of two odd primes. In particular, how stable would such a result be under a small perturbation of the primes?

Since we are concerned with only odd numbers, it is most appropriate to perturb into different odd numbers. Hence, we will consider a $\{-2, 0, 2\}$ perturbation of the set of odd primes P. An equivalent way of phrasing this problem is to consider the set $\bar{P} = \{(p-1)/2 \mid p \text{ is an odd prime}\}$. Goldbach's conjecture is now that $\bar{P}$ is a basis of order 2. (Or more precisely that $\bar{P} \cup \{0\}$ is such a basis, since zero must be an element of such a basis.) A $\{-1, 0, 1\}$-perturbation of $\bar{P}$ is equivalent to a $\{-2, 0, 2\}$-perturbation. We will show in the following theorem that there exists a $\{-1, 0, 1\}$-perturbation of $\bar{P}$ which is not even a subbasis of order 2. Moreover, the number of perturbed terms will have zero density in $\bar{P}$.

THEOREM 5. *There exists a $\{-2, 0, 2\}$-perturbation of the set P of odd primes, which is not a subbasis of order* 2.

Proof. As in previous theorems it suffices to construct an appropriate sequence $\{n_i\}$ such that $n_i \notin 2P'$, where P' is the perturbation of P. Since all elements of $2P'$ are even, we need only consider arithmetic progressions containing even terms.

P' will be constructed inductively. We will construct a sequence of sets

$\{P_k\}$, such that each P_k is a $\{-2, 0, 2\}$-perturbation of P_{k-1} such that no previously perturbed prime is perturbed again, and $n_k \notin 2P_k$.

Without loss of generality we may set $n_1 = 4$ and perturb 3 into 5 in P_1.

Now assume that P_k has been defined such that $n_1, n_2, ..., n_k \notin 2P_k$, and no prime $p > n_k/2$ has been perturbed. If A_{k+1} is the $(k+1)$st arithmetic progression, with common difference denoted by $2d$, then since there exist arbitrarily long strings of consecutive composite numbers, we can find such a string containing $[2n - n_k, 2n + 2dt]$ where $2n \in A_k$, for any desired t. Among the elements $2n, 2n + 2d, ..., 2n + 2dt$ in A_k choose $2n + 2ds$ such that $n + ds - 2$, $n + ds - 1$, $n + ds$ are all composite. (This is easily accomplished, for example, by the Chinese remainder theorem, since d is fixed and s can be arbitrarily large.)

Thus we have $2m = 2n + 2ds \in A_{k+1}$ such that all of the numbers $m-2$, $m-1$, m and $2m - n_k, ..., 2m$ are composite. Let $n_{k+1} = 2m$.

Consider all solutions of $p + q = 2m$ where p and q are primes, $p \leqslant q$. There are no such pairs where p is a previously perturbed prime, since if $p \leqslant n_k$ then $q = 2m - p$ is composite. For each such pair, perturb only the smaller prime p to $p + 2$ if $q - 2$ is composite or to $p - 2$ if $q + 2$ is composite. Clearly $q - 2, q, q + 2$ are not all primes. P_{k+1} is now the perturbation of P containing these newly perturbed primes and all previously perturbed ones. Also, no prime $p > n_{k+1}/2 = m$ has been perturbed.

Clearly, $n_1, ..., n_k \notin 2P_{k+1}$ since no element less than n_k has been changed. It remains only to show that $n_{k+1} = 2m \notin 2P_{k+1}$.

By construction, $2m$ is not a sum of two unperturbed primes. Also, $2m$ is not a sum of two perturbed primes. If p was perturbed into p', $p \leqslant m$, and by our choice of m; m, $m-1$, and $m-2$ are all composite. Hence $p \leqslant m-3$ and so when perturbed into p', $p' \leqslant m-1$. Thus the sum of any two perturbed primes is $\leqslant 2m - 2 < 2m$.

Finally, $2m$ is not the sum of one perturbed p' and one unperturbed prime q. If $p' + q = 2m$ and $p' = p + 2$, then $p + q_1 = 2m$ was an original solution and $q_1 - 2$ is composite by the construction. But $q_1 - 2 = 2m - p - 2 = 2m - p' = q$, so q cannot be a prime. Similarly, if $p' + q = 2m$ and $p' = p - 2$ then q cannot be a prime.

COROLLARY. *There exists a $\{-2, 0, 2\}$-perturbation of the set P of odd primes, which is not a subbasis of order* 2, *where the number of perturbed primes less than n is* $\ll n/\log n \log\log n$.

Proof. The proof of the preceding theorem can be modified by considering a number of cases, each quite trivial, so that the number $2m$ can be chosen to be any element of a long sequence of composite integers. Also, we may always perturb the larger of the pair p, q instead of the smaller.

In particular, once n_k has been chosen, we can choose N such that the interval $[N, 2N]$ has a string of $2t+1$ consecutive composite integers, where $2t+1 \leqslant c_1 \log N \leqslant 2t+3$, $t > n_k$, $c_1 > 0$. Suppose these numbers are $N_0, N_0+1, ..., N_0+2t < 2N$. Then $2m$ can be chosen to be any even number in the interval $[N_0+t, N_0+2t]$, since $t > n_k$. Choose $2m$ to be one having a minimal number of solutions of $p+q=2m$; $p, q \in P$.

Using the result of Hardy and Littlewood [3] that $\pi(x+y) - \pi(x) \ll y/\log y$, we estimate the number of times $p+q \in [N_0+t, N_0+2t]$. This is clearly bounded by

$$\sum_{p \leqslant \frac{1}{2}N_0+t} \pi(N_0+2t-p) - \pi(N_0+t-p)$$

$$\ll \sum_{p \leqslant \frac{1}{2}N_0+t} t/\log t \ll \frac{\frac{1}{2}N_0+t}{\log(\frac{1}{2}N_0+t)} \cdot \frac{t}{\log t}$$

$$\ll \frac{N_0}{\log \log N_0}.$$

Since there are $\gg \log N_0$ even numbers in $[N_0+t, N_0+2t]$, the number of solutions of $p+q=2m$ is $\ll N_0/(\log N_0 \log\log N_0)$.

Hence, this is also a bound on the number of perturbed primes $< 2N$. Since all of the newly perturbed primes may be taken to be greater than $\frac{1}{2}N_0$, the result follows.

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